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HOMOTOPY DG ALGEBRAS INDUCE HOMOTOPY BV ALGEBRAS

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Abstract

Let TA denote the space underlying the tensor algebra of a vector space A. In this short note, we show that if A is a differential graded algebra, then TA is a differential Batalin-Vilkovisky algebra. Moreover, if A is an A_∞ algebra, then TA is a commutative BV_∞ algebra.

1. Main Statement

Let (A, d_A) be a complex over a commutative ring R. Our convention is that d_A is of degree $+1$. The space $TA = \bigoplus_{n\geqslant 0} A^{\otimes n}$ is graded by declaring monomials of homogeneous elements $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ to be of degree $|a_1| + \cdots + |a_n| + n$.

There is a shuffle product $\bullet : TA \otimes TA \to TA$ generated by

$$(a_1 \otimes \cdots \otimes a_n) \bullet (a_{n+1} \otimes \cdots \otimes a_{n+m}) := \sum_{\sigma \in S(n,m)} (-1)^\kappa \cdot a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n+m)},$$

where $S(n,m)$ is the set of all (n,m)-shuffles, *i.e.* $S(n,m)$ is the set of all permutations $\sigma \in \Sigma_{n+m}$ with $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+m)$, (*cf.* [6]). Here $(-1)^\kappa$ is the Koszul sign, which introduces a factor of $(|a_i|+1)(|a_j|+1)$ whenever the elements a_i and a_j move past one another in a shuffle. Note that for degree zero elements of A, this Koszul sign is just $sgn(\sigma)$, the sign of the permutation σ. The shuffle product makes TA into a graded commutative associative algebra. Recall that TA is also a coalgebra under the usual tensor coproduct.

There is a differential $d : TA \to TA$ (of degree $+1$) given by extending the differential $d_A : A \to A$ as a coderivation of the tensor coproduct, see *e.g.* [7]:

$$d(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{|a_1|+\cdots+|a_{i-1}|+i-1} a_1 \otimes \cdots \otimes d_A(a_i) \otimes \cdots \otimes a_n$$

and together with the shuffle product, the triple $(TA, d, \bullet)$ is a differential graded commutative associative algebra.

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If $\mu_A : A \otimes A \to A$ is an associative product, then there is another differential $\Delta = \tilde{\mu}_A : TA \to TA$, of degree -1, given by extending the multiplication as a coderivation,

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{|a_1|+\cdots+|a_i|+i-1} a_1 \otimes \cdots \otimes \mu_A(a_i, a_{i+1}) \otimes \cdots \otimes a_n.$$

In Section 2 we show:

Theorem 1. *If (A, d_A, μ_A) is a differential graded algebra, then $(TA, d, \Delta, \bullet)$ defines a dBV algebra. The construction is functorial: If $f : A \to B$ is a morphism of differential associative algebras, then the induced map from TA to TB is a morphism of dBV algebras.*

Recall that a dBV algebra $(X, d, \Delta, \bullet)$ is a differential graded commutative associative algebra $(X, d, \bullet)$, with d of degree $+1$, and differential Δ of degree -1 such that d graded commutes with Δ (so that $d\Delta + \Delta d = 0$), and finally the deviation $\{,\}$ of Δ from being a derivation of $\bullet$,

$$\{x, y\} = (-1)^{|x|}\Delta(x \bullet y) - (-1)^{|x|}\Delta(x) \bullet y - x \bullet \Delta(y)$$

satisfies,

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\} \qquad \text{(Anti-symmetry)},$$
$$\{x \bullet y, z\} = x \bullet \{y, z\} + (-1)^{|y|(|z|+1)}\{x, z\} \bullet y \text{(Leibniz relation)}.$$

The Leibniz relation can be read as saying that bracketing with a fixed element (on the right) is a graded derivation of the product $\bullet$. These relations imply that bracketing with a fixed element on the left is also a graded derivation

$$\{x, y \bullet z\} = \{x, y\} \bullet z + (-1)^{(|x|+1)y} y \bullet \{x, z\}$$

and also imply that bracketing with a fixed element is a graded derivation of the bracket,

$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\} \qquad \text{(Jacobi identity)}.$$

A morphism of dBV algebras X and Y is a map $f : X \to Y$ that preserves the structures d, Δ, and $\bullet$.

Remark 1. In the special case where μ_A is graded commutative, Δ becomes a derivation of $\bullet$ and, thus, the bracket $\{,\}$ is zero. This is well known in the literature, see for example [5]. We were surprised we could not find in the literature the fact that TA becomes a dBV algebra when μ_A is not necessarily commutative. There is, however, a similar "Lie" version which is well known: the symmetric algebra of the underlying vector space of a Lie algebra is a BV algebra (see [8]).

Theorem 1 generalizes naturally. If $(A, \mu_1, \mu_2, \mu_3, \ldots)$ is an A_∞ algebra, then for each $k = 1, 2, \ldots$, the linear map $\mu_k : A^{\otimes k} \to A$ can be extended to a coderivation of degree $3 - 2k$ of the tensor coproduct $\Delta_{3-2k} : TA \to TA$. In Section 3 we show:

Theorem 2. *If $(A, \mu_1, \mu_2, \mu_3, \ldots)$ is an A_∞ algebra, then $(TA, \bullet, \Delta_1, \Delta_{-1}, \Delta_{-3}, \ldots)$ defines a commutative BV_∞ algebra.*

Remark 2. A commutative BV_∞ algebra, as defined by Kravchenko [4], is a generalization of a dBV algebra, and a special case of a BV_∞ algebra, as shown in [3]. (See also [1].) The precise definition is given in Section 3, where we show the requisite property that Δ_{3-2k} has operator-order k with respect to the shuffle product.

From a logical point of view, it is probably better to prove Theorem 2 first, from which Theorem 1 follows, see Remark 3 below. However, we prefer to give a direct proof of Theorem 1 using the traditional definition of a dBV algebra, making this an easy to read self-contained section. This also has the advantage of giving an explicit formula for the bracket $\{\, , \}$, and gives us the opportunity to illustrate explicitly how the signs are checked in this context.

2. Proof of the Theorem 1

The identities $d^2 = 0$, $\Delta^2 = 0$, $\bullet$ being associative and graded commutative, and d being a derivation of $\bullet$ are all straightforward. The (graded) anti-symmetry of the bracket follows formally from the (graded) symmetry of $\bullet$. The functoriality statement is immediate. It remains to show that the bracket $\{,\}$ satisfies the Leibniz relation.

We abbreviate $a_{i_1} \otimes \cdots \otimes a_{i_k}$ by $a_{i_1,\dots,i_k}$, and $\sigma^{-1}(i)$ by σ_i^{-1} for a permutation $\sigma \in \Sigma_k$. First, we may calculate the bracket as

$$\begin{aligned}\{a_{1,\dots,n}, a_{n+1,\dots,n+m}\} = &\sum_{\sigma\in S(n,m)} \pm\Delta(a_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}})\\ &- (\pm\Delta(a_{1,\dots,n}) \bullet a_{n+1,\dots,n+m}) - (\pm a_{1,\dots,n} \bullet \Delta(a_{n+1,\dots,n+m}))\end{aligned}$$

We claim that every term in the last two expressions cancels with precisely one term in $\sum_{\sigma\in S(n,m)} \pm\Delta(a_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}})$ so that $\{a_{1,\dots,n}, a_{n+1,\dots,n+m}\}$ equals

$$\sum_{\sigma\in S(n,m)} \sum_{j\in C_\sigma^{\{1,\dots,n\},\{n+1,\dots,n+m\}}} \pm a_{\sigma_1^{-1},\dots,\sigma_{j-1}^{-1}} \otimes \mu_A(a_{\sigma_j^{-1}}, a_{\sigma_{j+1}^{-1}}) \otimes a_{\sigma_{j+1}^{-1},\dots,\sigma_{n+m}^{-1}},$$

where the set $C_\sigma^{I,J}$ is defined, for a permutation $\sigma \in \Sigma_k$ and disjoint set of indices $I \cup J \subseteq \{1, \dots, k\}$ with $I \cap J = \varnothing$, by

$$C_\sigma^{I,J} = \{j \quad : \quad \sigma_j^{-1} \in I \text{ and } \sigma_{j+1}^{-1} \in J, \text{ or } \sigma_j^{-1} \in J \text{ and } \sigma_{j+1}^{-1} \in I\}.$$

In other words, μ_A is applied in the above sum whenever exactly one of the two elements $a_{\sigma_j^{-1}}$ and $a_{\sigma_{j+1}^{-1}}$ is taken from $a_1, \dots, a_n$, and the other element is taken from $a_{n+1}, \dots, a_{n+m}$. Since the correct terms appear exactly once, the only difficulty is to check the cancellation by signs, which we leave to the end of this section.

Assuming this, if we abbreviate the expression $a_{i_1,\dots,i_{j-1}} \otimes \mu_A(a_{i_j}, a_{i_{j+1}}) \otimes a_{i_{j+1},\dots,i_k}$ by $a^{(j,j+1)}_{i_1,\dots,i_k}$, then we can write,

$$\{a_{1,\dots,n}, a_{n+1,\dots,n+m}\} = \sum_{\sigma\in S(n,m)} \sum_{j\in C_\sigma^{\{1,\dots,n\},\{n+1,\dots,n+m\}}} \pm a^{(j,j+1)}_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}}$$

With this, we can check that $\{a_{1,\dots,n} \bullet a_{n+1,\dots,n+m}, a_{n+m+1,\dots,n+m+p}\}$ equals

$$\begin{aligned}
&= \sum_{\sigma\in S(n,m)} \pm\{a_{\sigma_1^{-1},\dots,\sigma_{n+m}^{-1}}, a_{n+m+1,\dots,n+m+p}\} \\
&= \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{1,\dots,n+m\},\{n+m+1,\dots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\dots,\rho_{n+m+p}^{-1}} \\
&= \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{1,\dots,n\},\{n+m+1,\dots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\dots,\rho_{n+m+p}^{-1}} \\
&\quad + \sum_{\rho\in S(n,m,p)} \sum_{j\in C_\rho^{\{n+1,\dots,n+m\},\{n+m+1,\dots,n+m+p\}}} \pm a^{(j,j+1)}_{\rho_1^{-1},\dots,\rho_{n+m+p}^{-1}} \\
&= a_{1,\dots,n} \bullet \{a_{n+1,\dots,n+m}, a_{n+m+1,\dots,n+m+p}\} \\
&\qquad\qquad \pm\{a_{1,\dots,n}, a_{n+m+1,\dots,n+m+p}\} \bullet a_{n+1,\dots,n+m},
\end{aligned}$$

where $S(n,m,p) \subseteq \Sigma_{n+m+p}$ consists of those permutations $\rho \in \Sigma_{n+m+p}$ that satisfy $\rho(1) < \cdots < \rho(n), \rho(n+1) < \cdots < \rho(n+m)$, and $\rho(n+m+1) < \cdots < \rho(n+m+p)$. By a careful consideration of the signs similar to the check below, it follows that the Leibniz identity holds.

Now, we check the sign mentioned above. If we shuffle $a_{n+1,\dots,n+j}$ past a_i, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, and then apply Δ, we obtain the term

$$a_{1,\dots,i-1} \otimes a_{n+1,\dots,n+j} \otimes \mu_A(a_i, a_{i+1}) \otimes a_{i+2,\dots n} \otimes a_{n+j+1,\dots,n+m}$$

with sign

$$(-1)^{(|a_i|+\cdots+|a_n|+n-i+1)(|a_{n+1}|+\cdots+|a_{n+j}|+j)+(|a_1|+\cdots+|a_{i-1}|+|a_{n+1}|+\cdots+|a_{n+j}|+|a_i|+(i-1+j))}$$

while in the other order, Δ then shuffle, we obtain the same term with sign

$$(-1)^{(|a_1|+\cdots+|a_i|+i+1)+(|\mu(a_i,a_{i+1})|+|a_{i+2}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j}|+j)}$$

and these agree. This special case implies the general case, for any shuffle, since a more general shuffle introduces the same additional sign in both cases.

Similarly, shuffling $a_{i+1,\dots,n}$ past a_{n+j+1} for $1 \leqslant i < n$ and $1 \leqslant j < m$, and then applying Δ, we obtain the term

$$a_{1,\dots,i} \otimes a_{n+1,\dots,n+j-1} \otimes \mu_A(a_{n+j}, a_{n+j+1}) \otimes a_{i+1,\dots,n} \otimes a_{n+j+2,\dots,n+m}$$

with sign

$$(-1)^{(|a_{i+1}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j+1}|+j+1)+(|a_1|+\cdots+|a_i|+|a_{n+1}|+\cdots+|a_{n+j}|+i+j+1)}$$

while in the other order we obtain the same term with sign

$$(-1)^{(|a_{n+1}|+\cdots+|a_{n+j-1}|+j-1)+(|a_{i+1}|+\cdots+|a_n|+n-i)(|a_{n+1}|+\cdots+|a_{n+j-1}|+|\mu(a_{n+j},a_{n+j+1})|+j)}$$

These differ by $(-1)^{|a_1|+\cdots+|a_n|+n}$, as expected. Again, this special case implies the general case, as before. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let $(X, \bullet)$ be a graded commutative associative algebra. An operator $\Delta : X \to X$ has operator-order n if and only if

$$\sum (-1)^{n+1-r+\kappa} \Delta\left(x_{i_1} \bullet \cdots \bullet x_{i_r}\right) \bullet x_{i_{r+1}} \bullet \cdots \bullet x_{i_{n+1}} = 0$$

where the sum is taken over nonempty subsets $\{i_1, \ldots, i_r : i_1 < \ldots < i_r\} \subseteq \{1, \ldots, n+1\}$ and $\{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_r\}$ has been ordered $i_{r+1} < \cdots < i_{n+1}$, and κ comes from the usual Koszul sign rule.

If Δ has operator-order one, then it is a derivation of $\bullet$. If Δ has operator-order two, then its deviation from being a derivation of $\bullet$, is a derivation of $\bullet$. This means that if we define $\{\,,\,\}$ to be the deviation of Δ from being a derivation of $\bullet$, then $\{\,,\,\}$ and $\bullet$ satisfy the Leibniz relation.

Remark 3. Using this fact, one can prove Theorem 1 without reference to the bracket—here is an outline: any map $\mu_A : A \otimes A \to A$ becomes an order 2 operator $\Delta : TA \to TA$ with respect to the shuffle product when it is lifted as a coderivation of the tensor coproduct (as we will show in the lemma below). It is straightforward to check that μ_A being associative implies that $\Delta^2 = 0$, since Δ^2 is the lift of the associator of μ_A to a coderivation. So, if (A, d_A, μ_A) is a differential graded algebra, with μ_A of degree zero, Δ has degree -1, and since d_A is a derivation of μ_A, then $d : TA \to TA$ and $\Delta : TA \to TA$ commute. That proves that $(TA, d, \Delta, \bullet)$ is a dBV algebra.

To generalize: a Kravchenko commutative BV_∞ algebra consists of a graded commutative differential graded algebra $(X, d, \bullet)$ and a collection $\{\Delta_k : X \to X\}_{k=1,-1,-3,-5,\ldots}$ of operators satisfying

- $\Delta_1 = d$,
- each Δ_{3-2k} has degree $3 - 2k$ and operator-order k,
- for each n, $\sum_{j+k=n} \Delta_j \Delta_k = 0$.

We use the degree convention in [4] but note that in [3] the opposite convention is used (there, d has degree -1 and the higher Δ operators have positive degree). As a special case, a dBV algebra is a Kravchenko commutative BV_∞ algebra with $\Delta_{-3} = \Delta_{-5} = \cdots = 0$.

To prove Theorem 2, assume that $(A, \mu_1, \mu_2, \mu_3 \ldots)$ is an A_∞ algebra. By definition of an A_∞ algebra, each μ_k lifts to a degree $3 - 2k$ coderivation $\Delta_{3-2k} : TA \to TA$, with $\Delta_1 = d$ and relations $\sum_{j+k=n} \Delta_j \Delta_k = 0$. Thus it only remains to prove, that each Δ_{3-2k} has order k with respect to the shuffle product $\bullet$. This follows from the following general lemma.

Lemma. *Let $f : A^{\otimes n} \to A$ be any linear map and let $F : TA \to TA$ be the lift of f to a coderivation. Then F has order n with respect to the shuffle product.*

Proof. Let $X^1, \ldots, X^{n+1}$ be monomials in TA. So, $X^i = a_1^i \otimes \cdots \otimes a_{s_i}^i$ with each $a_\ell^i \in A$. Then $(-1)^{n+1-r+\kappa} F(X^{i_1} \bullet \cdots \bullet X^{i_r}) \bullet X^{i_{r+1}} \cdots \bullet X^{i_{n+1}}$ consists of a sum of terms of the form

$$\pm \ldots \otimes f(a_{\ell_1}^{i'_1} \otimes \cdots \otimes a_{\ell_n}^{i'_n}) \otimes \ldots \text{ (the rest of the } a_\ell^i\text{'s are outside of } f), \qquad (1)$$

where f is applied to $a_{\ell_1}^{i'_1} \otimes \cdots \otimes a_{\ell_n}^{i'_n}$, and the remaining tensor products are applied outside of f. The list $\{i'_1, \ldots, i'_n\}$ may contain repetition, and we may order the list from smallest to largest without repetition as $\{i_1, \ldots, i_k\}$. Every term of the form (1) which contains only the indices $\{i_1, \ldots, i_k\}$ inside f, appears for each index set $J = \{j_1, \ldots, j_q\}$ with $\{i_1, \ldots, i_k\} \subseteq J \subseteq \{1, \ldots, n+1\}$ exactly once in the sum of $(-1)^{n+1-q+\kappa} F(X^{j_1} \bullet \cdots \bullet X^{j_q}) \bullet X^{j_{q+1}} \cdots \bullet X^{j_{n+1}}$. Now, for a fixed expression in Equation (1) induced by different index sets J, the only difference in the sign of (1) is a factor of $(-1)^q$, where $q = |J|$, and all other signs coincide for varying J. We thus need to show that summing $(-1)^{|J|}$ over all J with $\{i_1, \ldots, i_k\} \subseteq J \subseteq \{1, \ldots, n+1\}$ vanishes. Since there are exactly $n + 1 - k$ choose

$q-k$ such subsets J with q elements, we obtain that

$$\sum_J (-1)^{|J|} = \sum_{q=k}^{n+1} \binom{n+1-k}{q-k} \cdot (-1)^q = (-1)^k \cdot \sum_{q'=0}^{n+1-k} \binom{n+1-k}{q'} \cdot (-1)^{q'}$$
$$= (-1)^k \cdot (-1+1)^{n+1-k} = 0,$$

where we used the binomial theorem in the second to last equality. This completes the proof of the lemma. □

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