On the interplay between topology, geometry, and complex analysis on compact manifolds

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Structures of interest:

$$\text{Hermitian manifold} \mapsto \text{Complex manifold} \mapsto \text{Topological Manifold}$$

Questions:

1. What do various types of special Hermitian metrics tell us about the underlying compatible complex structure?
2. What do various types of complex structures tell us about the underlying topology?
3. In each case, we may study admissibility, or obstructions.

One formulation of the topology problem, in the words of Sullivan:

Prove *anything* about the topology
of compact complex manifolds in dimensions ≥ 6.

Open-ended, but still challenging. So, why is this compelling?

To quote Thurston:

> *"The product of mathematics is clarity and understanding. Not theorems, by themselves. Their real importance is not in their specific statements, but their role in challenging our understanding, presenting challenges that led to mathematical developments that increased our understanding."*

Deligne, Griffiths, Morgan, Sullivan (1975):

$$\exists \text{ Kähler metric} \implies \mathbb{C}\text{-structure satisfies } dd^c\text{-condition} \implies \text{topology is formal}$$

Among several results to explain today:

Theorem (Stelzig, W.)

$$\begin{aligned} \exists \ \textit{Vaisman metric} &\implies \mathbb{C}\textit{-structure satisfies } dd^c+3\textit{-condition} \\ \exists \ \textit{Vaisman metric} &\implies \textit{topology is almost formal } (\cong A_3\textit{-algebra}) \\ dd^c+3 \textit{ in low degree} &\implies \textit{restrictions on homotopy type in all dim's} \end{aligned}$$

To do: dd^c+3-condition, Ex's, Properties (blowups, deformations, etc.)

Review of dd^c-condition

For any complex manifold M, the differential forms A of M feature in the diagram

$$\begin{tikzcd}
 & (\operatorname{Im} d^c, d) \arrow[dr, "i"] & & \\
 & & (\operatorname{Ker} d^c, d) \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\
 & (A, d) \arrow[dl, "d^c"'] & & (H_{d^c}(A), d = 0) \\
(\operatorname{Im} d^c, d) & & &
\end{tikzcd}$$

where $d^c = J^{-1}dJ$. Fact: J integrable iff $[d^c, J_{der}] = 0 - d$.

Defn: The dd^c-condition holds iff the maps induce iso's on cohomology.

To see when this occurs, take (co)kernels: dd^c-condition holds iff $H_d(\operatorname{Im} d^c, d) = 0$.

Theorem

[The dd^c-condition, DGMS]
For any bounded bicomplex $(A, \partial, \bar\partial)$, with $d = \partial + \bar\partial$ and $d^c = i(\partial - \bar\partial)$,

the following are equivalent:

1. $H_d(\operatorname{Im} d^c, d) = 0$.
2. *For all $x \in A$, if $dx = 0$ and $x = d^c z$, then $x = dd^c w$ for some w.*
3. *E_1-degeneration and pure Hodge structure on $H_d(A)$.*
4. *The bicomplex $(A, \partial, \bar\partial)$ is a direct sum of*
   - *bicomplexes with only a single component, and $\partial = \bar\partial = 0$ (dots)*
   - *bicomplexes which are a square of isomorphisms (squares)*

dots:

$$\begin{tikzcd} 0 & \\ \mathbb{C} \arrow[u, "\bar\partial=0"] \arrow[r, "\partial=0"] & 0 \end{tikzcd}$$

squares:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\partial"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbb{C} \arrow[u, "\bar\partial"'] \end{tikzcd}$$

What other types of bicomplexes could occur?

Theorem (Stelzig, Khovanov-Qi)

Every (bounded) bicomplex $(A, \partial, \bar{\partial})$ decomposes as a direct sum of dots, squares, and zigzags.

In addition to dots and squares we have:

Length 3 zig-zags

$$
\begin{tikzcd}
\mathbb{C} & \\
\mathbb{C} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbb{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\partial"] & \mathbb{C} \\
& \mathbb{C} \arrow[u, "\bar\partial"']
\end{tikzcd}
$$

and more generally, odd-length zig-zags:

$$
\begin{tikzcd}
\mathbb{C} & & & & \\
\mathbb{C} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbb{C} & & & \\
& & \ddots & & \\
& & \mathbb{C} \arrow[r, "\partial"] & \mathbb{C} & \\
& & & \mathbb{C} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\partial"] & \mathbb{C} & & & \\
& \mathbb{C} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbb{C} & & \\
& & & \ddots & \\
& & & \mathbb{C} \arrow[r, "\partial"] & \mathbb{C} \\
& & & & \mathbb{C} \arrow[u, "\bar\partial"]
\end{tikzcd}
$$

Length 2 zig-zags

$$
\begin{tikzcd}
\mathbb{C} \\
\mathbb{C} \arrow[u, "\bar{\partial}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\partial"] & \mathbb{C}
\end{tikzcd}
$$

And more generally, even-length zig-zags:

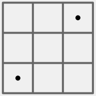
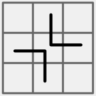
$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\partial"] & \mathbb{C} & & & \\
& \mathbb{C} \arrow[u, "\bar{\partial}"] & & & \\
& & \ddots & & \\
& & & \mathbb{C} \arrow[r, "\partial"] & \mathbb{C} & \\
& & & & \mathbb{C} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} & & & \\
\mathbb{C} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathbb{C} & & \\
& \ddots & & \\
& & \mathbb{C} \arrow[r, "\partial"] & \mathbb{C} \\
& & & \mathbb{C} \arrow[u, "\bar{\partial}"]
\end{tikzcd}
$$

Dictionary: "decomposition types $\leftrightarrow$ algebra"

1. No even zig-zags iff $H_{\bar\partial} \cong H_\partial \cong H_d$, i.e. E_1-degeneration.
2. No odd zig-zags of length > 1 iff every class in H_d has a unique representative of a single bi-degree (p, q) , i.e. *pure Hodge structure on* H_d.

Example

For $M = S^1 \times S^3$ with the complex structure of a Hopf manifold, the Frölicher spectral sequence degenerates, i.e. no even zig-zags, and

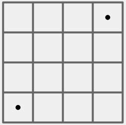
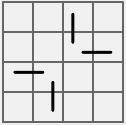
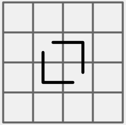
$A(M) \simeq$  $\oplus$  .

Similarly, for all complex surfaces: there are only dots, squares and length 3 zig-zags.

Unlike compact complex surfaces, in $\dim \geq 6$ there can be even length zig-zags, and odd zig-zags of length > 3.

Example

For $M = S^3 \times S^3$ with the Calabi-Eckmann complex structure , $h^{0,1}_{\bar{\partial}}(M) \neq 0$ but $H^1_d(M) = 0$.
It follows from calculations of Angella and Tomassini that

$A(M) \simeq$  $\oplus$  $\oplus$  .

Theorem (Stelzig, W., the $dd^c + 3$-condition)

For any bounded bicomplex A, the following are equivalent:

1. *The bicomplex $(A, \partial, \bar{\partial})$ decomposes as a direct sum of dots, squares and length 3 zigzags.*
2. *The Frölicher (row- and column-) spectral sequences degenerate at E_1, and the purity defect is equal to 1.*
3. *The following holds, for all $k \geq 0$: For all $x \in A^k$, if $x = dy$ and $x = d^c z$, then $x = dw$ with $w \in \operatorname{Ker} d^c$.*
4. *The following numerical equality holds:*

$$\sum_k \dim H^k(\operatorname{Ker} d^c) + \dim H^k(A/\operatorname{Im} d^c) = 2\sum_k b_k.$$

Purity defect "pdef" measures (roughly) how many distinct bi-degrees might be needed to represent a given class in $H_d(A)$.

Easiest definition: $\mathrm{pdef} = k$ iff the longest odd length zig-zag has length $2k + 1$.

New character in the story: $H_d(A/\operatorname{Im} d^c)$

For any complex manifold there is a diagram

$$\begin{tikzcd}
 & (\operatorname{Ker} d^c, d) \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\
(A, d) \arrow[dr, "p"'] & & (H_{d^c}, d = 0) \arrow[dl, "j"] \\
 & (A/\operatorname{Im} d^c, d) &
\end{tikzcd}$$

and a short exact sequence of complexes:

$$0 \longrightarrow (\operatorname{Ker} d^c, d) \xrightarrow{i+\pi} (A, d) \oplus (H_{d^c}, 0) \xrightarrow{p-j} (A/\operatorname{Im} d^c, d) \longrightarrow 0,$$

So, every complex manifold induces a long exact sequence in cohomology:

$$\cdots \xrightarrow{\delta_{k-1}} H^k(\mathrm{Ker}\, d^c) \xrightarrow{i+\pi} H^k_d \oplus H^k_{d^c} \xrightarrow{p-j} H^k(A/\mathrm{Im}\, d^c) \xrightarrow{\delta_k} H^{k+1}(\mathrm{Ker}\, d^c) \longrightarrow \cdots$$

Lemma [Stelzig, W.] $\quad dd^c + 3 \iff \delta \equiv 0$.

Corollary:

$$\sum_k \dim H^k(\mathrm{Ker}\, d^c) + \dim H^k(A/\mathrm{Im}\, d^c) \geq 2 \sum_k b_k.$$

with equality iff $dd^c + 3$.

Aside on Hopf's Problem:

δ is an isomorphism iff the manifold is a homology sphere (rationally).
Only known example of such a complex manifold is S^2.
Are there others?
Note: Albanese and Milivojević's construction yields many potential examples.

In the category of complex manifolds, the $dd^c + 3$-condition satisfies:

1. A blow-up of a manifold M along a smooth center $Z \subseteq M$ is $dd^c + 3$ if and only if both M and Z are $dd^c + 3$.
2. A product is $dd^c + 3$ if and only if one factor is a $dd^c + 3$-manifold and one is a dd^c-manifold.
3. The target of a holomorphic surjection $f : M \to N$ with M a $dd^c + 3$-manifold and $\dim M = \dim N$ is again a $dd^c + 3$-manifold.
4. Projectivized holomorphic vector bundles are $dd^c + 3$-manifolds if and only if the base of the bundle is a $dd^c + 3$-manifold.
5. Any sufficiently small deformation of a $dd^c + 3$-manifold is again a $dd^c + 3$-manifold.

More generally, in words: if there are no even zig zags, the length of the longest odd zig-zag can only go down in a small deformation of the complex structure.

Sketch of proof of stability:
$dd^c + 3 \iff \dim H_{\bar\partial} = \dim H_d$ and $\mathrm{pdef} = 1$.

Classical result: $\dim H_{\bar\partial} = \dim H_d$ is stable, since $\dim H_{\bar\partial} \geq \dim H_d$, and $\dim H_{\bar\partial}$ is semi-continuous in families.

Suffices to show: $\dim H_{\bar\partial} = \dim H_d \implies \mathrm{pdef} \in \mathbb{N}$ is semi-continuous ,

Idea: express pdef in terms of a vector-bundle built from intersections of various filtrations of cohomology. In fact,

$$\mathrm{pdef} = \left| \max_{p,q,k \geq 0} \left\{ p + q - k \,\middle|\, F^p H_d^k(M;\mathbb{C}) \cap \bar{F}^q H_d^k(M;\mathbb{C}) \neq 0 \right\} \right|$$

and $\{F^p H_d^k(M_t)\}$ form a complex vector subbundle of the vector bundle $\{H_d^k(M_t)\}$ over the base $t \in \Delta$ of a family M_t.

Examples of $dd^c + 3$ manifolds

- all compact complex surfaces
- higher Hopf manifolds $S^1 \times S^{2n-1}$
- certain twistor spaces
- simply connected examples (Kasuya, Stelzig)
- many nilmanifolds

Theorem (Stelzig, W.)

If a compact complex manifold admits a Vaisman metric then the underlying complex structure is $dd^c + 3$.

A metric ω is called Vaisman if $d\omega = \theta \wedge \omega$, with θ parallel.

In fact, one can compute which zigzags appear in which positions within the bicomplex of forms of a Vaisman manifold.

Consider $\mathcal{H}_B$, the subspace of d-harmonic *basic forms*, which are invariant under the group action generated by dual holomorphic vector fields X_θ and $X_{J\theta}$.

Write: $\theta = \theta^{1,0} + \theta^{0,1}$.

Theorem (Tsukada 1994, Ishida and Kasuya 2019)

The subspace

$$\mathcal{H}_B \otimes \Lambda\langle \theta^{0,1}, \theta^{1,0} \rangle \subseteq A(V)$$

is a d-subcomplex and inclusion induces an isomorphism.

Behavior is similar to Kähler manifold, having a Lefschetz decomposition given by the operator L given by wedging with $\omega_0 = d^c\theta$. One computes the bicomplex using the primitive decomposition.

Namely, $\mathcal{H}_B \otimes \Lambda\langle \theta^{0,1}, \theta^{1,0}\rangle$ decomposes as a direct sum of tensor products of primitive forms (dots) with bicomplexes of the form

$$
\begin{array}{ccc}
\langle \theta^{0,1} \rangle & \oplus & \langle \theta^{1,0}\theta^{0,1} \rangle \\
\oplus & & \oplus \\
\mathbb{C} & \oplus & \langle \theta^{1,0} \rangle
\end{array}
$$

and

$$
\begin{tikzcd}
 & & & & & & \langle \theta^{0,1}\omega_0^{n-k} \rangle & \langle \theta^{0,1}\theta^{1,0}\omega_0^{n-k} \rangle \\
 & & & & & & \langle \theta^{1,0}\theta^{0,1}\omega_0^{n-k-1} \rangle \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \langle \theta^{1,0}\omega_0^{n-k} \rangle \\
 & & & & & \iddots & & \\
 & & & \langle \theta^{0,1}\omega_0^{j+1} \rangle \arrow[r, "\partial"] & \langle \omega_0^{j+2} \rangle & & & \\
 & & & \langle \theta^{1,0}\theta^{0,1}\omega_0^{j} \rangle \arrow[u, "\bar\partial"] \arrow[r, "\partial"'] & \langle \theta^{1,0}\omega_0^{j+1} \rangle \arrow[u, "\bar\partial"'] & & & \\
 & & \iddots & & & & & \\
\langle \theta^{0,1} \rangle \arrow[r, "\partial"] & \langle \omega_0 \rangle & & & & & & \\
\mathbb{C} & \langle \theta^{1,0} \rangle \arrow[u, "\bar\partial"'] & & & & & &
\end{tikzcd}
$$

Finally, I want to indicate how the rational homotopy type restricts the types of bicomplexes that can occur, for a complex manifold structure with a given underlying topology.

Example

Consider a filiform nilmanifold $M = G/\Gamma$ where Γ is a lattice in the simply connected Lie group G associated with the left-invariant forms

$$\Lambda(\eta^1, ..., \eta^6) \quad d\eta^1 = d\eta^2 = 0, \; d\eta^k = \eta^1\eta^{k-1} \text{ for } k = 3, ..., 6.$$

Non-formal nilmanifold with $b_1 = 2$ and trivial ring structure on H^1.

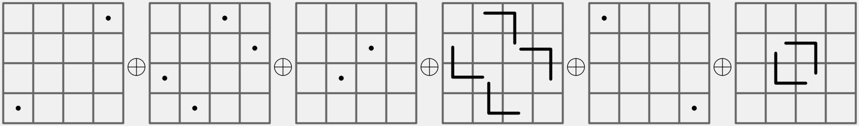
1. Admits an almost complex structure (e.g. put $J\eta^{2k} = \eta^{2k-1}$).
2. Does not admit left-invariant complex structures (Goze-Remm, 2002)
3. Does admit J_t where $N(J_t) \to 0$ (Fernandez, Shin, W.).
4. It is unknown whether it admits any complex structure.

Which bi-complexes could occur for a hypothetical complex structure on M?

$dd^c + 3 \implies$ topological restrictions

Question

Is it possible that M admits a complex structure with the following bicomplex decomposition:



This would yield the correct Betti numbers and satisfy the $dd^c + 3$ condition.

No, in fact...rational homotopy theory shows $dd^c + 3$ *must* fail:

Theorem (Stelzig, W.)

No compact 6-manifold with the homotopy type of this nilmanifold can support such a complex structure.

A sketch of the argument for this claim, which generally gives rational homotopy obstructions to $dd^c + 3$-complex structures, even in low degree.

The diagram

$$\begin{tikzcd}
 & (\mathrm{Ker}\, d^c, d) \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\
(A, d) \arrow[dr, "p"'] & & (H_{d^c}, d = 0) \arrow[dl, "j"] \\
 & (A/\mathrm{Im}\, d^c, d) &
\end{tikzcd}$$

has additional *symmetries* when A is differential forms on compact $2n$-manifold:

1. (left-right symmetry) The left and right side induce maps of same rank cohomology groups, for all k.
2. (top-bottom duality) $H^k(\mathrm{Ker}\, d^c) \cong (H^{2n-k}(A/\mathrm{Im} d^c))^\vee$, for all k.

In particular, the top maps induce an iso in top degree.

The filiform nilmanifold is *far* from being formal: all the topology is generated from forms of degrees ≤ 1, even the fundamental class.

But, the cohomology ring in degrees ≤ 1 is trivial, so does not create the fundamental class.

Taking a minimal model $\mathcal{M}$ for the algebra $\operatorname{Ker} d^c$,

$$\begin{tikzcd} & \mathcal{M} \arrow[d] & \\ & (\operatorname{Ker} d^c, d) \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\ (A, d) & & (H_{d^c}, d = 0) \end{tikzcd}$$

and using

$$dd^c + 3\text{-condition} \iff i \oplus \pi \text{ is injective on cohomology},$$

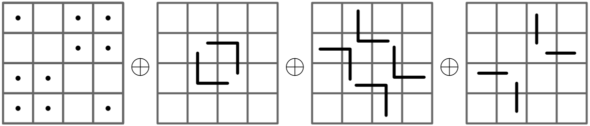
one can play the two sides of the diagram against each other, and contradict the left-right symmetry of induced maps in top degree cohomology.

$dd^c + 3 \implies$ topological restrictions

An example which does have a complex structure:
Let $M = G/\Gamma$ be a nilmanifold with structure equations

$$\begin{aligned} d\eta^3 &= \eta^1\eta^2 & \qquad d\eta^4 &= \eta^1\eta^3 \\ d\eta^5 &= \eta^2\eta^3 & \qquad d\eta^6 &= \eta^1\eta^4 + \eta^2\eta^5. \end{aligned}$$

Then $b_1(M) = 2$, and $\eta^1\eta^2 = d\eta^3$, so the product $\cup : H^1 \times H^1 \to H^2$ is trivial. There are two left-invariant complex structures, both degenerate at E_2 but not E_1, and satisfy purity in degree 1, and pdef ≤ 1. For each

$A(M) \simeq$ 

Similar arguments using rational homotopy show this manifold cannot have a complex structure that is $dd^c + 3$ with pure Hodge in degree 1, left invariant or not. In particular, the "lines" in the last diagram cannot be dropped.

Thank you!

Reference: "A dd^c-type condition beyond the Kähler realm," Stelzig, Wilson
J. Inst. Math. Jussieu, 2024.

Relation with Bott-Chern and Appeli cohomologies:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 & \\
\cdots H_d^k \oplus H_{d^c}^k \arrow[r] & H^k(A/\operatorname{Im} d^c) \arrow[r, "\delta_k"] \arrow[d, hook, "\psi"'] & H^{k+1}(\operatorname{Ker} d^c) \arrow[r] \arrow[u] & H_d^{k+1} \oplus H_{d^c}^{k+1} \cdots \\
 & H_A^k(A) \arrow[r, "d"] \arrow[d, two heads] & H_{BC}^{k+1}(A) \arrow[u, two heads, "\phi"'] & \\
 & H_{\llcorner}^k(A) \arrow[r, "d=0"] \arrow[d] & H_{\urcorner}^{k+1}(A) \arrow[u, hook] & \\
 & 0 & 0 \arrow[u] &
\end{tikzcd}
$$