Small Nijenhuis tensors on almost complex manifolds with no complex structure

Scott Wilson

joint with Luis Fernandez, Tobias Shin

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  In general, the fiber is non-compact space of dimension $n(2n) = 2n^2$.

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There are <u>many</u> other equivalent ways to formulate integrability.

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  - There are almost complex 6-manifolds for which there is no known complex structure (one example below).

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- Remark on motivation from h-principle:
  - Can express an integrable structure as solution to "closed differential relation".
  - Via h-principle techniques one can try to deform a "formal" solution to a "genuine" one.

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Nilmanifold review

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Let's do an example...

Filiform manifolds, dimension 4

Let

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If $\{x_1, x_2, x_3, x_4\}$ is the dual basis,

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Now use these to compute the real cohomology.

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\begin{tikzcd}[column sep=small]
3 & & -x_1x_2x_3 & -x_1x_2x_4 & x_1x_3x_4 & x_2x_3x_4 & \\
2 & -x_1x_2 & -x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 \arrow[ulll] & x_3x_4 \arrow[ulll] \\
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\end{tikzcd}$$

Therefore, by Poincaré Duality, the Betti numbers of $\Gamma\backslash G$ are

$$b_1 = b_2 = b_3 = 2$$

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Corollary

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- Note that $\Gamma\backslash G$ is symplectic: $x_1x_4 + x_2x_3$ is closed and non-degenerate.

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Note that $\lim_{t\to\infty} J_t$ does not exist:

$$\lim_{t\to\infty} J_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ +\infty & -1 & 0 & 0 \\ 0 & 0 & -1-\sqrt{2} & 0 \\ 0 & 0 & +\infty & 1+\sqrt{2} \end{bmatrix}.$$

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- So, any such filiform manifold $\Gamma\backslash G$ has no complex structure, but has a family J_t with arbitrarily small Nuijenhuis tensor.
- We have no conceptual explanation for this!

Fernández and Gray manifold, dimension 4

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where $\mathfrak{g}(k)$ the Lie algebra of the simply connected solvable (non-nilpotent) Lie group $G(k)$ given by matrices of the form

$$\begin{bmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $x, y, z \in \mathbb{R}$ and $k \neq 0$.

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Theorem (Hattori): The real cohomology of any completely solvmanifold can be computed from the cohomology of the Lie algebra $\mathfrak{g}$.

- Solvable Lie groups may not admit a lattice. There is no general criteria.
- But $G(k)$ does admit a lattice (many in fact; details omitted).
- Let $M^4(k) = (\Gamma \backslash G(k)) \times S^1$.
- The Lie algebra is completely solvable, i.e. the adjoint action ad_X has real eigenvalues for all X.

Theorem (Hattori): The real cohomology of any completely solvmanifold can be computed from the cohomology of the Lie algebra $\mathfrak{g}$.

In this case, one can compute the Betti numbers of $M^4(k)$ as before:

$$b_1 = b_2 = b_3 = 2.$$

Fernández and Gray manifold, dimension 4

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- are symplectic (and therefore almost Kähler).
- satisfy all known cohomological properties of Kähler manifolds.
- are formal (in fact the same minimal model as $S^1 \times S^1 \times S^2$).
- but neverththeless are not Kähler, as they are not even complex.

For any $k \neq 0$, consider the family of linear almost complex structures on $\mathfrak{g}$ defined in the ordered basis $\{X_1, \ldots, X_4\}$ by:

For any $k \neq 0$, consider the family of linear almost complex structures on $\mathfrak{g}$ defined in the ordered basis $\{X_1, \ldots, X_4\}$ by:

$$J_t = \begin{bmatrix} \frac{-2}{kt^2} & \frac{-1}{\sqrt{3}} & -\frac{6+\sqrt{3}kt^2+2k^2t^4}{3k^2t^3} & \frac{6-\sqrt{3}kt^2+2k^2t^4}{3k^2t^5} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{3}kt} - \frac{2t}{3} & \frac{\sqrt{3}-2kt^2}{3kt^3} \\ \frac{1}{t} & \frac{1}{t} & \frac{1}{\sqrt{3}} + \frac{1}{kt^2} & \frac{-1}{kt^4} \\ -t & t & \frac{-1}{k} & \frac{-1}{\sqrt{3}} + \frac{1}{kt^2} \end{bmatrix}.$$

One can show that $J_t^2 = -\mathrm{Id}$.

Fernández and Gray manifold, dimension 4

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N_t(X_1, X_2) &= \frac{-2k}{\sqrt{3}t}(X_1 + X_2) + \frac{2k}{t^2}X_3 \\
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N_t(X_1, X_4) &= \frac{2}{kt^6}X_1 - \frac{1}{t^5}X_3 + \frac{1}{\sqrt{3}t^4}(X_2 - X_1) + \frac{1}{t^3}X_4 + \frac{2k}{3t^2}(X_1 + X_2) \\
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N_t(X_3, X_4) &= \frac{4}{k^2t^7}X_1 - \frac{2}{kt^6}X_3 + \frac{2}{\sqrt{3}kt^5}(X_2 - X_1) + \frac{2}{t^4}\left(\frac{1}{\sqrt{3}}X_3 + \frac{1}{k}X_4\right) \\
&\quad + \frac{2}{3t^3}(X_1 + X_2) - \frac{4k}{3t^2}X_3 + \frac{4k}{3\sqrt{3}t}(X_1 + X_2).
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Again, we have no conceptual explanation for this.

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So, there is no complex structure on M^6 induced as a nilmanifold.
Unknown whether M^6 admits a complex structure.

Filiform manifolds, dimension 6

We give an example of a 1-parameter family J_t, of left-invariant almost-complex structures on any M^6, such that the Nijenhuis-tensor $N_t := N(J_t)$ satisfies $N_t \to 0$ as $t \to \infty$.

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$$J_t = \begin{bmatrix} \frac{t^2-1}{\sqrt{3}\left(t^2+1\right)} & -\frac{1}{t^3} & 0 & 0 & 0 & 0 \\ \frac{4t^3\left(t^4+t^2+1\right)}{3\left(t^2+1\right)^2} & -\frac{t^2-1}{\sqrt{3}\left(t^2+1\right)} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{t} & 0 & 0 \\ -\frac{2t^3}{\sqrt{3}\left(t^2+1\right)} & \frac{1}{t^2} & -\frac{4t}{3} & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{5t^4+4t^2+1}{\sqrt{3}t^3\left(2t^2+1\right)} & \frac{8t^2(t^2+1)}{3\sqrt{3}(2t^2+1)} & -\frac{4t^5}{6t^4+9t^2+3} & \frac{1}{\sqrt{3}} & \frac{t^2+1}{t^3} \\ \frac{4t^3\left(t^4+t^2+1\right)}{3\sqrt{3}\left(2t^4+3t^2+1\right)} & \frac{2}{3} & -\frac{16t^5\left(2t^4+2t^2+1\right)}{9\left(t^2+1\right)^2\left(2t^2+1\right)} & -\frac{8t^4}{\sqrt{3}\left(6t^2+3\right)} & -\frac{4t^3}{3\left(t^2+1\right)} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

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N_t(X_i, X_j) &= 0 \ \text{ if } i, j \geq 3.
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Therefore $N_t \to 0$ in the C^0-norm on $M^6 = \Gamma \backslash G$.

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Thank you!

L. Fernandez, T. Shin, S. O. Wilson, *"Almost complex manifolds with small Nijenhuis tensor"*, arXiv:2103.06090.

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According to Hasegawa's classification of compact complex 4-dimensional solvmanifolds, any such solvmanifold $\Gamma\backslash G$ does not have a complex structure.

Write $A = VDV^{-1}$, with D diagonal having entries $e^{\lambda_1}, e^{\lambda_2}, e^{-(\lambda_1+\lambda_2)}$, and V is a matrix whose columns are the respective eigenvectors.

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In the standard basis $\{E_i\}$ of $\mathbb{R} \times \mathbb{R}^3$, with $E_1 = X_1$, the Lie algebra of G has non-zero brackets determined by

$$[E_1, E_i] = (\log A)\, E_i,$$

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Equivalently, the Lie bracket $\{\,,\,\} := V^{-1} \circ [\,V(-), V(-)]$, satisfies

$$\begin{aligned} \{X_1, X_2\} &= \lambda_1 X_2 \\ \{X_1, X_3\} &= \lambda_2 X_3 \\ \{X_1, X_4\} &= -(\lambda_1 + \lambda_2) X_4. \end{aligned}$$

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Note that the first Betti number of $\Gamma \backslash G$ is equal to one unless any of λ_1, λ_2, or $(\lambda_1 + \lambda_2)$ are zero.

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Dimension 4

To give a 1-parameter family of almost complex structures J_t on $(\mathfrak{g}, [\,,\,])$ with $N_{J_t} \to 0$, it suffices to give almost complex structures K_t on $(\mathfrak{g}, \{\,,\,\})$ in the basis $\{X_1, X_2, X_3, X_4\}$, with $N_{K_t} \to 0$, for then we may define $J_t := V^{-1} K_t V$.

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In the ordered basis $\{X_1, \ldots, X_4\}$, let

$$K_t = \begin{bmatrix} 1 & 1/t & \frac{2(\lambda_1+2\lambda_2)}{(\lambda_1-\lambda_2)t} & 0 \\ -\frac{2(2\lambda_1+\lambda_2)t}{\lambda_1-\lambda_2} & -\frac{2\lambda_1+\lambda_2}{\lambda_1-\lambda_2} & -\frac{2(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}{(\lambda_1-\lambda_2)^2} & \frac{\lambda_1+2\lambda_2}{(\lambda_1-\lambda_2)t} \\ t & 1/2 & \frac{\lambda_1+2\lambda_2}{\lambda_1-\lambda_2} & \frac{-1}{2t} \\ 0 & t & \frac{2(2\lambda_1+\lambda_2)t}{\lambda_1-\lambda_2} & 0 \end{bmatrix}$$

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One can show that $K_t^2 = -\mathrm{Id}$.

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$$N_t(X_2, X_3) = \left(\frac{2\,(\lambda_1 + 2\lambda_2)}{t^2}\right) X_1 - \left(\frac{2(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)}{(\lambda_1 - \lambda_2)t}\right) X_2 + \left(\frac{\lambda_1 + 2\lambda_2}{t}\right) X_3$$

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