

SMALL NIJENHUIS TENSORS ON ALMOST COMPLEX MANIFOLDS WITH NO COMPLEX STRUCTURE

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joint with Luis Fernandez, Tobias Shin



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In general, the fiber is non-compact space of dimension $n(2n) = 2n^2$.

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There are many other equivalent ways to formulate integrability.

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Buchdahl's study of the intersection pairing on $\text{Ker}(\bar{\partial}\partial)$.
- In real dimension 6: It is unknown whether there are any further obstructions to the existence of an integrable structure.
 - There are almost complex 6-manifolds for which there is no known complex structure (one example below).

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- Remark on motivation from h -principle:
 - Can express an integrable structure as solution to “closed differential relation”.
 - Via h -principle techniques one can try to deform a “formal” solution to a “genuine” one.

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Let's do an example...

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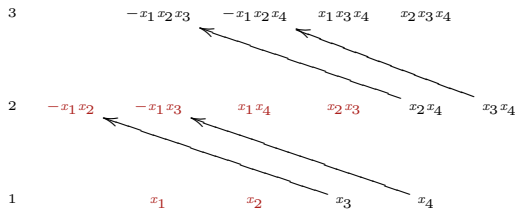
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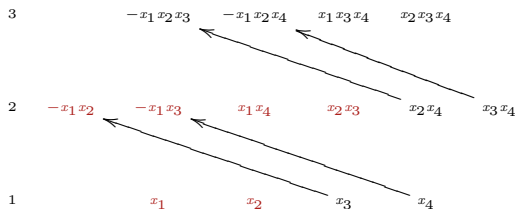
Now use these to compute the real cohomology.

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Therefore, by Poincaré Duality, the Betti numbers of $\Gamma \backslash G$ are

$$b_1 = b_2 = b_3 = 2$$

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- But $b_2 = 2 \neq 4$, so $\Gamma \backslash G$ is not a torus.
- Note that $\Gamma \backslash G$ is symplectic: $x_1 x_4 + x_2 x_3$ is closed and non-degenerate.

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Note that $\lim_{t \rightarrow \infty} J_t$ does not exist:

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- So, any such filiform manifold $\Gamma \setminus G$ has no complex structure, but has a family J_t with arbitrarily small Nijenhuis tensor.
- We have no conceptual explanation for this!

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where $\mathfrak{g}(k)$ the Lie algebra of the simply connected solvable (non-nilpotent) Lie group $G(k)$ given by matrices of the form

$$\begin{bmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $x, y, z \in \mathbb{R}$ and $k \neq 0$.

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In this case, one can compute the Betti numbers of $M^4(k)$ as before:

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- are symplectic (and therefore almost Kähler).
- satisfy all known cohomological properties of Kähler manifolds.

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- satisfy all known cohomological properties of Kähler manifolds.
- are formal (in fact the same minimal model as $S^1 \times S^1 \times S^2$).
- but nevertheless are not Kähler, as they are not even complex.

Fernández and Gray manifold, dimension 4

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For any $k \neq 0$, consider the family of linear almost complex structures on $\mathfrak{g}$ defined in the ordered basis $\{X_1, \dots, X_4\}$ by:

$$J_t = \begin{bmatrix} \frac{-2}{kt^2} & \frac{-1}{\sqrt{3}} & -\frac{6+\sqrt{3}kt^2+2k^2t^4}{3k^2t^3} & \frac{6-\sqrt{3}kt^2+2k^2t^4}{3k^2t^5} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{3}kt} - \frac{2t}{3} & \frac{\sqrt{3}-2kt^2}{3kt^3} \\ \frac{1}{t} & \frac{1}{t} & \frac{1}{\sqrt{3}} + \frac{1}{kt^2} & \frac{-1}{kt^4} \\ -t & t & \frac{-1}{k} & \frac{-1}{\sqrt{3}} + \frac{1}{kt^2} \end{bmatrix}.$$

One can show that $J_t^2 = -\text{Id}$.

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Again, we have no conceptual explanation for this.

Methodology

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Unknown whether M^6 admits a complex structure.

We give an example of a 1-parameter family J_t , of left-invariant almost-complex structures on any M^6 , such that the Nijenhuis-tensor $N_t := N(J_t)$ satisfies $N_t \rightarrow 0$ as $t \rightarrow \infty$.

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$$J_t = \begin{bmatrix} \frac{t^2-1}{\sqrt{3}(t^2+1)} & -\frac{1}{t^3} & 0 & 0 & 0 & 0 \\ \frac{4t^3(t^4+t^2+1)}{3(t^2+1)^2} & -\frac{t^2-1}{\sqrt{3}(t^2+1)} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{t} & 0 & 0 \\ -\frac{2t^3}{\sqrt{3}(t^2+1)} & \frac{1}{t^2} & -\frac{4t}{3} & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{5t^4+4t^2+1}{\sqrt{3}t^3(2t^2+1)} & \frac{8t^2(t^2+1)}{3\sqrt{3}(2t^2+1)} & -\frac{4t^5}{6t^4+9t^2+3} & \frac{1}{\sqrt{3}} & \frac{t^2+1}{t^3} \\ \frac{4t^3(t^4+t^2+1)}{3\sqrt{3}(2t^4+3t^2+1)} & \frac{2}{3} & -\frac{16t^5(2t^4+2t^2+1)}{9(t^2+1)^2(2t^2+1)} & -\frac{8t^4}{\sqrt{3}(6t^2+3)} & -\frac{4t^3}{3(t^2+1)} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

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 N_t(X_i, X_j) &= 0 \text{ if } i, j \geq 3.
 \end{aligned}$$

Therefore $N_t \rightarrow 0$ in the C^0 -norm on $M^6 = \Gamma \backslash G$.

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Thank you!



L. Fernandez, T. Shin, S. O. Wilson, “*Almost complex manifolds with small Nijenhuis tensor*”, arXiv:2103.06090.

Dimension 4

Consider the Lie group $\mathbb{R}^3 \rtimes_{\phi} \mathbb{R}$, where $\phi : \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^3)$, and $\mathbb{R}$ and $\mathbb{R}^3$ have the standard additive structures.

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$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & 8 \end{bmatrix}$$

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Extended: $\phi : \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^3)$ by defining $\phi_t = \exp(t \log A)$.

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To obtain lattice: choose $\phi(1) = A \in SL(3, \mathbb{Z})$, and restrict to the case that A has three real, positive, distinct eigenvalues.

Example:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & 8 \end{bmatrix}$$

for any integer k with $6 \leq k \leq 15$.

Extended: $\phi : \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^3)$ by defining $\phi_t = \exp(t \log A)$.

Let $\Gamma = \mathbb{Z}^3 \rtimes_{\phi} \mathbb{Z}$ be lattice in $G = \mathbb{R}^3 \rtimes_{\phi} \mathbb{R}$.

According to Hasegawa's classification of compact complex 4-dimensional solvmanifolds, any such solvmanifold $\Gamma \backslash G$ does not have a complex structure.

Dimension 4

Write $A = VDV^{-1}$, with D diagonal having entries e^{λ_1} , e^{λ_2} , $e^{-(\lambda_1+\lambda_2)}$, and V is a matrix whose columns are the respective eigenvectors.

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Note that the first Betti number of $\Gamma \backslash G$ is equal to one unless any of λ_1 , λ_2 , or $(\lambda_1 + \lambda_2)$ are zero.

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To give a 1-parameter family of almost complex structures J_t on $(\mathfrak{g}, [\cdot, \cdot])$ with $N_{J_t} \rightarrow 0$, it suffices to give almost complex structures K_t on $(\mathfrak{g}, \{\cdot, \cdot\})$ in the basis $\{X_1, X_2, X_3, X_4\}$, with $N_{K_t} \rightarrow 0$, for then we may define $J_t := V^{-1}K_t V$.

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In the ordered basis $\{X_1, \dots, X_4\}$, let

$$K_t = \begin{bmatrix} 1 & 1/t & \frac{2(\lambda_1+2\lambda_2)}{(\lambda_1-\lambda_2)t} & 0 \\ -\frac{2(2\lambda_1+\lambda_2)t}{\lambda_1-\lambda_2} & -\frac{2\lambda_1+\lambda_2}{\lambda_1-\lambda_2} & -\frac{2(2\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)}{(\lambda_1-\lambda_2)^2} & \frac{\lambda_1+2\lambda_2}{(\lambda_1-\lambda_2)t} \\ t & 1/2 & \frac{\lambda_1+2\lambda_2}{\lambda_1-\lambda_2} & -\frac{1}{2t} \\ 0 & t & \frac{2(2\lambda_1+\lambda_2)t}{\lambda_1-\lambda_2} & 0 \end{bmatrix}$$

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One can show that $K_t^2 = -\text{Id}$.

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Then, for each non-zero and distinct λ_1, λ_2 , $N_t \rightarrow 0$ as $t \rightarrow \infty$.

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